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OPTIMAL CONDITIONS OF DECELERATING THE ROTATIONAL MOTION OF A SYMMETRIC BODY

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The problem of determining the optimum operating conditions of a torque-producing jet engine during deceleration of the angular velocity of a symmetric rigid body, executing originally free motion in space about its own center of inertia, is mathematically analyzed. Braking within minimum time and braking at minimum consumption of the working medium are calculated, at controlling moments of limited magnitude. Optimum phase trajectories of the body, for both cases, are derived.

The control of the rotary motion of a rigid body by means of a system of torque-producing jet engines reduces to selecting the conditions of programing the thrust of such engines in conformity with the conditions of a specific problem. Below, we will examine the problem of determining the optimum operating conditions of engines during deceleration of the angular velocity of a symmetric rigid body executing originally free motion in space about its own center of inertia. We will examine two regimes of deceleration: braking in minimum time (neglecting consumption of the working medium) and braking at minimum consumption of working medium (neglecting time). Although, in individual particular cases, both these regimes may coincide, in the general case they are different and thus require a separate investigation.

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* Numbers in the margin indicate pagination in the original foreign text.

We will assume in the analysis that the jet engines create controlling moments about the principal axes of inertia of the body and that, during consumption of the working medium, the moments of inertia of the body remain essentially constant (just as the directions of the principal axes in the body). The controlling moments are considered to be limited in magnitude.

As a result of the investigation it was found that, in the first case, all three controlling moments act in reverse up to complete arrest of the body and, in the second case, the transverse moments are included alternately, whereas the longitudinal moment (directed along the axis of symmetry of the body) remains included up to complete elimination of the longitudinal component of the angular velocity of the body. The construction of the phase trajectories is given in the problem of eliminating precessional motion of the body at constancy of its longitudinal velocity component.

1. Problem of Braking in Minimum Time

Let us examine the problem of finding the optimum conditions of decelerating a symmetric rigid body with respect to speed of response. Assuming, for definitiveness, that the polar moment of inertia of the body C is greater than its equatorial moment of inertia A, let us write the system of differential equations of motion of the body in the form of

$$\dot{\omega_x} + \varepsilon\omega_y\omega_z = \frac{m_x}{A}, \qquad \dot{\omega_y} - \varepsilon\omega_x\omega_z = \frac{m_y}{A}, \qquad \dot{\omega_z} = \frac{m_z}{C}. \qquad \left(\varepsilon = \frac{C-A}{A} > 0\right) \tag{1.1}$$

The law of change of the controlling moments m_x, m_y, and m_z (relative to the principal axes of inertia x, y, z) must be determined such that the angular velocity components ω_x, ω_y, ω_z (hereafter playing the role of phase coordinates) will acquire the prescribed final values in minimum time. In the case of complete braking of the body the final values of the angular velocity should become equal to zero, whereas in the case of incomplete braking, for example when only the precessional motion of the body is eliminated, only the transverse com- /726 ponents $\omega_x(T)$ and $\omega_y(T)$, where T is the time of completing the process, should become equal to zero. In view of the linearity of the controlling moments relative to the derivatives in equations (1.1) it is convenient to use the principle of the maximum for setting up the variational problem (Bibl.1). Let us set up the function H of the problem under consideration

$$H = \sum_k p_k \dot{\omega}_k = p_x \left(\frac{m_x}{A} - \varepsilon \omega_y \omega_z \right) + p_y \left(\frac{m_y}{A} + \varepsilon \omega_x \omega_z \right) + p_z \frac{m_z}{C} \tag{1.2}$$

and write the system of equations for the phase impulses

$$p_k = - \partial H / \partial \omega_k .$$

In a developed form, it will be

$$\dot{p}_x + \varepsilon p_y \omega_z = 0, \qquad \dot{p}_y - \varepsilon p_x \omega_z = 0, \qquad \dot{p}_z - \varepsilon (p_x \omega_y - p_y \omega_x) = 0 . \tag{1.3}$$

Without specifying the boundary conditions of the problem for the time being, let us construct the necessary integrals of eqs.(1.1) and (1.3) and investigate the general character of the optimal control conditions. For this purpose, using the principle of the maximum, let us establish the optimum law of change of the controls m_k. Since the function H depends linearly on the controls, it reaches its maximum at values of controls equal to their limiting values; if the multiplier p_k on m_k is positive then the control will be at its upper limit; if this multiplier is negative, it will be at the lower limit. Thus, at $p_k \not\equiv 0$ the optimum conditions of the change of moments m_k will be on-off* and

* If $P_k \equiv 0$, special conditions arise which may differ from on-off. These conditions are discussed in greater detail in Lee's work (Bibl.3).

will be determined by the following relations:

$$m_k(t) = \max m_k \text{ at } p_k(t) > 0, \qquad m_k(t) = \min m_k \text{ at } p_k(t) < 0 . \tag{1.4}$$

If the limits of the change of m_k are symmetric with respect to zero, then

$$m_k(t) = \max|m_k| \operatorname{sgn} p_k(t) . \tag{1.5}$$

Here $\max |m_k|$ is the peak value of the k-th control. Hereafter in this problem, m_k will mean the quantity determinable by eq.(1.5).

2. Integrations of the Equations of Optimum Motion

To obtain the solution of the system of equations (1.1) and (1.3) in which the controls are determined according to eqs.(1.5) we will first isolate the following subsystem containing the variables p_x, p_y, and ω_z

$$p_x^{\cdot} + \varepsilon p_y \omega_z = 0, \qquad p_y^{\cdot} - \varepsilon p_x \omega_z = 0, \qquad \omega_z^{\cdot} = m_z / C . \tag{2.1}$$

According to eq.(1.5), at each individual section of motion the moment m_z is constant so that the solution of the last of the written equations in the section will be

$$\omega_z = \omega_{z0} + C^{-1} m_z t . \tag{2.2}$$

Thus, the z-th component of the angular velocity of the body under optimal braking conditions is a step function whose points of discontinuity according to eq.(1.5), correspond to the roots of the function p_z.

Let us introduce into the calculation the complex function /727

$$p = p_x + i p_y . \tag{2.3}$$

From the first two equations of the system (2.1), we can obtain

$$p^{\cdot} - i\varepsilon\omega_z p = 0 . \tag{2.4}$$

The solution of this equation has the form

$$p = p_0 \exp\left(i\varepsilon \int_0^t \omega_z \, dt\right). \tag{2.5}$$

Taking into account eq.(2.2) we find

$$\varepsilon \int_0^t \omega_z \, dt = \frac{\varepsilon m_z}{2C} t^2 + \varepsilon \omega_{z0} t = \lambda (\omega_z^2 - \omega_{z0}^2) \qquad \left(\lambda = \frac{\varepsilon C}{2m_z}\right). \tag{2.6}$$

Then, in place of eq.(2.5) we will have

$$p \exp(-i\lambda\omega_z^2) = \text{const} \tag{2.7}$$

The expression for the function p can also be represented in the form of

$$p = P \exp[i(\lambda\omega_z^2 + \alpha)], \tag{2.8}$$

where P and α are real constants of integration determinable from the boundary conditions of the problem. Hence, for p_x and p_y we have

$$p_x = P \cos(\lambda\omega_z^2 + \alpha), \qquad p_y = P \sin(\lambda\omega_z^2 + \alpha). \tag{2.9}$$

Multiplying the second equation of the system (1.1) by i and adding it to the first, we obtain the equation for the complex transverse angular velocity of the body:

$$\omega^{\bullet} - i\varepsilon\omega_z\omega = A^{-1}m \qquad (\omega = \omega_x + i\omega_y,\ m = m_x + im_y). \tag{2.10}$$

The general integral of this equation according to Lur'ye (Bibl.2) has the form

$$\omega = \left[\omega_0 + \frac{m}{A}\int_0^t \exp\left(-i\varepsilon\int_0^t \omega_z dt\right) dt\right] \exp\left(i\varepsilon\int_0^t \omega_z dt\right). \tag{2.11}$$

Taking into account expression (2.6) and then changing from the variable t to ω_z, an integration will yield

$$\omega = \omega_0 \exp[i\lambda(\omega_z^2 - \omega_{z0}^2)] + \frac{m(\varepsilon+1)}{m_z}\left(\frac{\pi}{2|\lambda|}\right)^{1/2}[C - C_0 -$$

$$- i \operatorname{sgn} \lambda (S - S_0)] \exp (i\lambda\omega_z^2) \tag{2.12}$$
$$(S = S(\omega_z \sqrt{|\lambda|}), \quad S_0 = S(\omega_{z0} \sqrt{|\lambda|}) .$$

Here C(w) and S(w) are the Fresnel integrals,

$$C(w) = \sqrt{\frac{2}{\pi}} \int_0^w \cos v^2 dv, \qquad S(w) = \sqrt{\frac{2}{\pi}} \int_0^w \sin v^2 dv . \tag{2.13}$$

The integral (2.12) can also be written as

$$\omega \exp(- i\lambda\omega_z^2) - \frac{m(\varepsilon + 1)}{m_z} \sqrt{\frac{\pi}{2|\lambda|}} [C - i \operatorname{sgn} \lambda S] = D = \text{const} . \tag{2.14}$$

To determine the function p_z, we can dispense with integration of the /728 corresponding differential equation: this function enters the integral of the H problem which can be written in the form of

$$m_z C^{-1} p_z - \varepsilon\omega_z (p_x\omega_y - p_y\omega_x) + A^{-1} (p_x m_x + p_y m_y) = 1 . \tag{2.15}$$

The right-hand side of this integral remains a unit constant during the entire braking process. This is explained by the fact that here the relay controls m_k stand as multipliers in front of the corresponding functions p_k which vanish at the instant of switching the controls.

3. Investigation of the Solution

The integrals found for the phase coordinates and pulses contain a sufficient number of constants for solving the two-point boundary-value problem under consideration. As the three constants determining the initial position of the phase point in phase space we can select, for example, the initial velocity values ω_{x0}, ω_{y0}, ω_{z0}. Then, to satisfy the end conditions of the problem (for example to obtain $\omega_x(T) = \omega_y(T) = \omega_z(T) = 0$ in the case of complete braking) this leaves the quantities P, α, T, where T is the time of the braking process and P and α are constants of integration entering into eqs.(2.9). It follows from the requirement of continuity of p_x and p_y at the points of switching the moments m_z that their amplitude P and phase $(\lambda\omega_z^2 + \alpha)$ should also be continuous. This, in turn, indicates that the constant P retains its value during the entire braking process, whereas the constant α changes stepwise by a magnitude $\pm 2|\lambda|\omega_z^2$ at the points of switching the moment m_z, remaining continuous at the points of switching the moments m_x and m_y.

Taking into account the expressions (2.9) for p_x and p_y and also the relations (1.5), we can rewrite the integral of eq.(2.15) as

$$
\begin{aligned}
&|p_z m_z| C^{-1} + PA^{-1} [|m_x \cos(\lambda\omega_z^2 + \alpha)| + |m_y \sin(\lambda\omega_z^2 + \alpha)|] - \\
&\quad - \varepsilon P \omega_z [\omega_y \cos(\lambda\omega_z^2 + \alpha) - \omega_x \sin(\lambda\omega_z^2 + \alpha)] = 1 .
\end{aligned}
\tag{3.1}
$$

Hence, the obtained integral determines only the modulus of the function p_z but not its sign. We can express $|p_z|$ also as an explicit function ω_z, after eliminating ω_x and ω_y by means of eq.(2.14). However, for numerical calculations it is apparently more convenient to use eq.(3.1) directly.

The laws of switching the moments m_x and m_y are given by the following expressions:

$$
m_x = |m_x|_{\max} \operatorname{sgn} \cos(\lambda\omega_z^2 + \alpha), \qquad m_y = |m_y|_{\max} \operatorname{sgn} \sin(\lambda\omega_z^2 + \alpha) . \tag{3.2}
$$

In this case, switching of the moments m_x and m_y occurs respectively at points determinable by the equations

$$
\lambda\omega_z^2 + \alpha = n\pi + {}^1/_2\pi, \qquad \lambda\omega_z^2 + \alpha = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots) . \tag{3.3}
$$

The phase trajectory of the system is determined by eq.(2.14), whose right-hand side changes stepwise at the points of switching the moments m_x, m_y, m_z. We will find the magnitude of this jump at the point of switching the moment m_z. By virtue of the continuity of ω and ω_z, we then obtain /729

$$
\Delta D(\Delta m_z) = D_2 - D_1 = 2i\omega \sin \lambda\omega_z^2 + \frac{2m(\varepsilon + 1)}{(m_z)_1} \left(\frac{\pi}{2|\lambda|}\right)^{1/2} C . \tag{3.4}
$$

At the point of switching the moment m at ΔD, we will have

$$\Delta D\,(\Delta m) = D_2 - D_1 = \frac{(m_1 - m_2)(\varepsilon + 1)}{m_z}\left(\frac{\pi}{2|\lambda|}\right)^{1/2}(C - i\,\mathrm{sgn}\,\lambda\,S)\,. \tag{3.5}$$

Here the subscript 1 denotes the value before switching and the subscript 2, after switching; D is a constant determinable by the integral (2.14). Equations (3.4) and (3.5) permit joining the phase trajectory at the points of switching the controls in a numerical calculation of the braking process. Equation (2.14) is equivalent to the two real equations:

$$\begin{aligned} &\omega_x \cos \lambda\omega_z^2 + \omega_y \sin \lambda\omega_z^2 = \mathrm{Re}\, D + \frac{\varepsilon+1}{m_z}\left(\frac{\pi}{2|\lambda|}\right)^{1/2}(m_x C + m_y\, \mathrm{sgn}\,\lambda\, S) \\ &\omega_x^2 + \omega_y^2 = |D|^2 + \frac{2(\varepsilon+1)}{m_z}\left(\frac{\pi}{2|\lambda|}\right)^{1/2}[C\,\mathrm{Re}\,(D\bar{m}) - \mathrm{sgn}\,\lambda\, S\, \mathrm{Im}\,(D\bar{m})] + \\ &\qquad + \frac{|m|^2(\varepsilon+1)^2\pi}{2m_z^2|\lambda|}(C^2 + S^2)\,. \end{aligned} \tag{3.6}$$

Hence, the phase trajectory is the line of intersection of the ruled helicoidal surface with a surface of revolution, i.e., a certain spatial spiral-like curve of variable radius and variable pitch twisting about the axis ω_z.

At sufficiently large values of ω_z, at $\lambda\omega_z^2 \gg 1$, we can simplify the expression for the angular velocity ω as a function of ω_z. Substituting the asymptotic expansion for the Fresnel integrals

$$C\,(\omega_z\sqrt{|\lambda|}) \approx \frac{1}{2} + \frac{\sin|\lambda|\,\omega_z^2}{\omega_z\sqrt{2\pi|\lambda|}}, \qquad S\,(\omega_z\sqrt{|\lambda|}) \approx \frac{1}{2} - \frac{\cos|\lambda|\,\omega_z^2}{\omega_z\sqrt{2\pi|\lambda|}} \tag{3.7}$$

into eq.(2.14) and omitting the unessential constants, we obtain

$$\left(\omega - \frac{im}{\omega_z(C-A)}\right)\exp\,(-\,i\lambda\omega_z^2) = \mathrm{const}\,. \tag{3.8}$$

Multiplying both sides of eq.(3.8) by their conjugate quantities, we find

$$\left(\omega_x + \frac{m_y}{\omega_z(C-A)}\right)^2 + \left(\omega_y - \frac{m_x}{\omega_z(C-A)}\right)^2 = \mathrm{const}\,. \tag{3.9}$$

As we see, at sufficiently large values of ω_z, the projection of the phase trajectory onto the plane $\omega_x\omega_y$ will consist of a sequence of arcs similar to arcs of circles.

Thus, the derived solution for the coordinates ω_k and for the pulses p_k enables us, in principle, to calculate the nonsingular optimal deceleration conditions of a symmetric body. However, two unknown constants P and α, determinable from the boundary conditions of the problem, enter these integrals. Therefore, for a specific calculation of the braking process it is more convenient to examine it in the opposite direction, assuming, for example, that at the initial instant $\omega_x = \omega_y = \omega_z = 0$. Then, having assigned the values P and α we can /730 calculate the entire course of the phase trajectory up to a certain point $\omega_x(T)$, $\omega_y(T)$, $\omega_z(T)$. After this, changing the initial values of P and α we can "hit" a point in the phase space as close as desired to the given point. We note that, from the formula for the initial value of the modulus p_{z0}

$$|p_{z0}| = \frac{C}{|m_z|}\left[1 - \frac{P}{A}\left(|m_x \cos\alpha_0| + |m_y \sin\alpha_0|\right)\right] \tag{3.10}$$

follows the condition

$$0 \leqslant P < \frac{A}{|m_x \cos\alpha_0| + |m_y \sin\alpha_0|}. \tag{3.11}$$

Depending on the selection of the sign of the quantity p_{z0} we obtain two phase trajectories, symmetric with respect to the plane $\omega_z = 0$. For definitiveness, we can set $p_{z0} > 0$ and, consequently, $m_{z0} > 0$, $\lambda_0 > 0$. The signs of the controlling moments m_{x0} and m_{y0}, depending upon the value of the angle α_0, are determined from eqs.(3.2).

Figure 1 shows one of the phase trajectories of the problem under consideration plotted by means of the described method, i.e., actually for the process of acceleration of a body from zero initial velocity up to some final value. By virtue of the reversibility of the solution of the variational problem, this trajectory will be optimal also for the process of braking a body from this point $\omega_x(T)$, $\omega_y(T)$, $\omega_z(T)$ in the phase space up to the point $\omega_x = \omega_y = \omega_z = 0$.

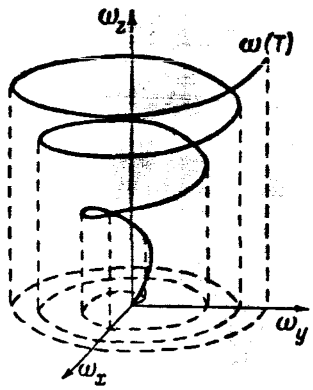

Fig.1

4. Braking at Minimum Consumption of Mass

Assuming that the controlling moments m_k are created by a jet engine with a constant exhaust velocity and that the engines, creating the moments m_x and m_y, are identical, we will take as the functional of the problem the following integral

$$M = \int_0^T (\mu |m_x| + \mu |m_y| + \nu |m_z|)\, dt \qquad (4.1)$$

where M is a quantity proportional to the mass consumed for deceleration and μ and ν are positive coefficients. Thus, in this problem we must supplement the system of equations of motion (1.1) by an additional equation for the function M

$$M^{\cdot} = \mu |m_x| + \mu |m_y| + \nu |m_z| \qquad (4.2)$$

and seek the conditions for switching the controlling moments m_k such that M(T) will be minimal. However, eq.(4.2) is inconvenient in that the moduli of functions with alternating signs enter into it. To eliminate this drawback, we will assume that the values of m_k can be only positive or equal to zero

$$0 \leqslant m_k \leqslant |m_k|_{\max} \tag{4.3}$$

and will assign their sign in eqs.(1.1) by introducing additional equations of u_k, whose entire set of permissible values is limited only by two points $u_k = \pm 1$. Then, in place of the systems (1.1) and (4.2) we will have /731

$$\begin{aligned} &\omega_x^{\cdot} + \varepsilon\omega_y\omega_z = \frac{m_x u_x}{A}, \quad \omega_y^{\cdot} - \varepsilon\omega_x\omega_z = \frac{m_y u_y}{A} \\ &\omega_z^{\cdot} = \frac{m_z u_z}{C}, \quad M^{\cdot} = \mu m_x + \mu m_y + \nu m_z . \end{aligned} \tag{4.4}$$

It is obvious that the differential equations and thus also the integrals for the functions p_x and p_y remain as before. The integral determining p_z will vary, since the form of the function H will vary where we must evidently introduce the new quantity p_M which is conjugate to the variable M. It follows from the formulation of the problem that the quantity p_M will be a unit constant equal to -1 during the entire braking process. With consideration of this function, we can write H in the following form:

$$\begin{aligned} H = {} & m_x\left(\frac{p_x u_x}{A} - \mu\right) + m_y\left(\frac{p_y u_y}{A} - \mu\right) + \\ & + m_z\left(\frac{p_z u_z}{C} - \nu\right) - \varepsilon\omega_z(p_x\omega_y - p_y\omega_x) . \end{aligned} \tag{4.5}$$

From the condition of the maximum of this function with respect to the controls m_k and u_k it follows that

$$u_k = \operatorname{sgn} p_k, \qquad m_k = |m_k|_{\max} \operatorname{Sg}(p_k u_k - I_k) . \tag{4.6}$$

Here,

$$I_x = I_y = \mu A, \quad I_z = \nu C, \quad \operatorname{Sg} w = 1 \text{ at } w > 0, \quad \operatorname{Sg} w = 0 \text{ at } w < 0 .$$

As before, p_z is determined directly from the integral (4.5) which, with consideration of eq.(4.6), can be written in the following manner:

$$m_x\left(\frac{|p_x|}{A}-\mu\right)+m_y\left(\frac{|p_y|}{A}-\mu\right)+m_z\left(\frac{|p_z|}{C}-\nu\right)=\omega_z\dot{p_z}. \tag{4.7}$$

Since the left-hand side of this equation in optimal conditions cannot be negative, then

$$\operatorname{sgn}\dot{p_z}=\operatorname{sgn}\omega_z . \tag{4.8}$$

It follows from eq.(4.7) that, at the end of the braking process when $\omega_x = \omega_y = \omega_z = 0$, the left-hand side of the equation should also be equal to zero, i.e., the conditions

$$|p_k(T)|\leqslant I_k \tag{4.9}$$

should be satisfied.

In cases in which, over a certain section, the condition

$$|p_k| = I_k \tag{4.10}$$

is satisfied, singular control conditions may arise at which the magnitude of the controlling moments may assume not only their own boundary values.

It is evident that a singular regime with respect to the transverse moments m_x and m_y can occur only simultaneously for both and only when the longitudinal velocity ω_z is completely eliminated, since otherwise, according to eq.(2.4), the functions p_x and p_y cannot become constants satisfying the condition (4.10). With respect to the longitudinal moment m_z, the occurrence of a singular condition is characterized by the conditions $\omega_z \neq 0$, $|p_z| = \nu C$. It follows from this that $\dot{p}_z = 0$, i.e., either $\omega_x = \omega_y = 0$ or $p_x\omega_y = p_y\omega_x$. /732

The first case corresponds to the regime of pure rotation of a body about a longitudinal axis where, of course, the law of variation of the moment m_z does not affect the overall consumption of the working medium.

The second case, as is readily demonstrated, can occur only in passive zones with respect to the transverse moment, i.e., in the sections where m = 0. We note that in the general case in passive zones where m = 0, the derivative $\dot{p}_z$

assumes a constant value [this is easily demonstrated by means of eqs.(2.5) and (2.11)] so that the function p_z varies linearly in these time intervals.

It follows from the form of the integral (4.7) that, during the entire braking process, the function p_z varies monotonically, continuously increasing or continuously decreasing depending upon the sign of the initial velocity ω_{z0}. Assuming that, at the initial instant, we have $\omega_{z0} > 0$, then, according to eq.(4.8), $\dot{p}_{z0} > 0$ and the function p_z will increase. In this case, it is necessary that the condition $p_{z0} < -\nu C$ be satisfied since otherwise, by virtue of eqs.(4.4) and (4.6), the angular velocity ω would begin to increase continuously and no deceleration will be accomplished.

Analogously, if $\omega_{z0} < 0$, then the initial value of p_{z0} should be greater than νC. Since the above statements remain valid for any instant of time, we can conclude that the sign of the function p_z should always be opposite to the sign of ω_z, i.e.,

$$u_z = -\operatorname{sgn} \omega_z \ . \tag{4.11}$$

Thus, deceleration of longitudinal angular velocity takes place by switching the moment m_z in a direction opposite to the direction of ω_z. If the magnitude of this moment remains constant at all times, then the duration of its action τ will be determined by the formula

$$\tau = \frac{|\omega_{z0}|\, C}{|m_z|} \tag{4.12}$$

and

$$|p_z(\tau)| = \nu C \ . \tag{4.13}$$

The law of variation of ω_z will then be linear so that, assuming $\omega_{z0} > 0$ for definitiveness, we will have

$$\omega_z = \omega_{z0} - m_z C^{-1} t \ . \tag{4.14}$$

In view of this, until $t < \tau$, i.e., until the velocity ω_z vanishes, we can as before use ω_z in place of the time t as the independent variable. Thus, in the interval $0 \le t \le \tau$ the integrals (2.8) and (2.14) remain valid, and we need only introduce the equations of u_k before the corresponding moments m_k. In this case, of course, it must be remembered that, unlike the problem of the speed of response, passive zones appear in the equations of m_x and m_y, i.e., sections in which $m_k = 0$ $(k = x, y)$. /733

The position of these zones is determined by the condition

$$|p_x| \leqslant \mu A, \qquad |p_y| \leqslant \mu A . \tag{4.15}$$

From the requirement that these conditions at the final instant of time must be satisfied simultaneously, the amplitude P will be expressed by the inequality

$$\mu A \leqslant P \leqslant \mu A \sqrt{2} . \tag{4.16}$$

5. Deceleration of Transverse Velocity

The particular case of this problem is the problem of eliminating precessional motions of a body, i.e., of removing the transverse components of the

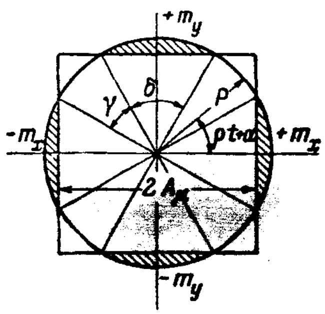

Fig.2

angular velocity of a symmetric body ω_x and ω_y at $m_z \equiv 0$. In these conditions, the form of the solution will differ since $\omega_z = \omega_{z0}$ = const; consequently, in place of the system (1.1) we will have a system of linear equations with constant coefficients. Designating

$$\varepsilon\omega_{z0} = \rho \tag{5.1}$$

for the complex angular velocity ω and the function $p = p_x + ip_y$ we will have the following integrals

$$\left(\omega - \frac{im}{\rho A}\right) e^{-i\rho t} = \text{const}, \qquad pe^{-i\rho t} = \text{const} . \tag{5.2}$$

Hence, the phase trajectory in the plane $\omega_x \omega_y$ will be a curve comprised of arcs of circles. Unlike the speed-of-response problem, here arcs with a center at the origin of the coordinates corresponding to the zones where $m_x = m_y = 0$ will also become a part of these curves.

To plot the phase trajectory of the system we will first construct a diagram in the form of a circle of radius P and a square with a side $2\mu A$, as shown in Fig.2. Then the terminus of the vector $p = Pe^{i(\rho t + \alpha)}$, upon moving along this circle, will osculate the zones corresponding to different values of the controls m_x and m_y and, obviously, the hatched segments will correspond to sections of inclusion of the moments m_x and m_y while the unhatched segments correspond to passive zones, i.e., to sections $m_x = m_y = 0$. We easily see that the width of the passive zone γ, just as the width of each of the active zones δ, depends on the value of the ratio $\mu A/P$ and will obey the following relations:

$$\cos {}^1\!/_2\delta = \mu A / P, \qquad \gamma + \delta = {}^1\!/_2\pi . \tag{5.3}$$

By means of this diagram, we can construct a family of phase trajectories proceeding from the origin of the coordinates. It is evident that the first section should be active, for example $m_x \neq 0$, and for definitiveness we will assume that $\cos \alpha_0 > \mu A/P$ so that $u_x = 1$.

Then, according to eq.(5.2) the equation of the initial segment of the phase curve will become

$$\omega_x^2 + \left(\omega_y - \frac{m_x}{\rho A}\right)^2 = \left(\frac{m_x}{\rho A}\right)^2. \tag{5.4}$$

As soon as the complex vector p in Fig.2 turns through an angle 1/2 δ, i.e., as soon as the complex vector $\omega - i(m_x/\rho A)$ in the phase plane rotates relative /734 to the point a_1 through an angle $1/2\,\delta - \alpha$, the moment m_x will be excluded and, furthermore, the phase curve will be an arc of circle described from the origin of the coordinates to the central angle γ. Then, the moment m_y will be included, the phase trajectory will pass along the arc of circle described from the center at the point $a_2(\omega_y = 0, \omega_x = -m_y/\rho A)$, and the length of the arc obviously will make the angle δ. After this, the moment m_y is excluded and the trajectory becomes a segment of the arc γ described from the origin of the coordinates.

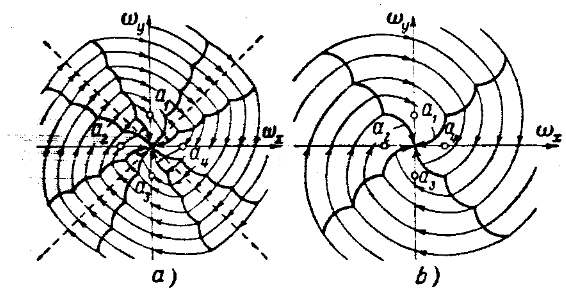

Fig.3

Continuing this process we can show that the family of phase trajectories in the axes $\omega_x\omega_y$ will have the form shown in Fig.3a. As we see, the entire phase plane is divided into eight sectors by switching lines made up of arcs of circles, the length of each of which in angular measure is equal to δ. Four of these sectors correspond to the active zones of braking and the four others to the passive zones, where motion by inertia takes place. We note that the width of the passive zone γ in this problem, where control of longitudinal velocity is not provided for, can be arbitrary within the limits $0 \leq \gamma \leq 1/2\pi$. The width of this zone affects only the duration of the braking process and does not affect the consumption of mass. In the limiting case, where $\gamma \to 0$, the passive zones disappear and the pattern of the phase plane takes the form shown in Fig.3b. It should be mentioned that the pattern of the phase trajectories actually does not differ here from that of the phase trajectories in the analogous problem of the speed of response, with the exception of the rotation of the entire phase plane through an angle $1/4\pi$.

Thus, a comparison of the conditions optimal with respect to speed of response and consumption of the mass of the working medium clearly shows that the qualitative difference is that, in the latter, the transverse controlling moments m_x and m_y do not act simultaneously but are cut in alternately; furthermore, the longitudinal moment m_z does not reverse during the deceleration and always has a sign opposite to that of the longitudinal component of the angular velocity ω_z.

In conclusion, it should be pointed out that the practical realization of the examined optimal conditions is extremely difficult in view of the complex character of the switching surfaces located in the phase space of the system. However, the solutions do permit finding the maximum possible speed of response or economy of operation of a control system and thus to evaluate, from this viewpoint, the quality of an arbitrarily selected nonoptimum control regime.

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